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On Extension of Functions with Zero Trace on a Part of the Boundary

GEORGE MEJAK

Department of Mathematics and Mechanics, University of Ljubljana, 61111 Ljubljana, Slovenia

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INTRODUCTION

Let u be an arbitrary function from the Sobolev space $W^{m,p}(\Omega)$ and let us consider a question of extending u to a function which belongs to $W^{m,p}(\mathbb{R}^n)$. In the simplest case, when u has zero trace on the boundary Γ (we assume enough regularity of Ω so that the notion of the trace is meaningful), extension by zero to the whole of $\mathbb{R}^n$ rises to the required extension. In the general case, when u does not have zero trace, there are several well known extension theorems; see, e.g., [1, Chap. 4], which under certain conditions on Ω guarantees the existence of the extension. However, those theorems does not control propagation of the support; in other words, if u has the zero trace on a part of the boundary $M \subset \Gamma$, then the support of the extension might touch M from the outside of Ω.

It is the aim of the paper to construct an extension with controllable propagation of the support. In particular, it is shown that for any finite cone C such that $(x+C) \cap \Omega = \varnothing$ for all $x \in M$, there exists, under certain conditions on M, an extension operator $\Pi: W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ such that if a function $u \in W^{m,p}(\Omega)$ has zero trace on M, then

$$(\operatorname{supp} \Pi u) \cap (x+C) = \varnothing \qquad \text{for all} \quad x \in M. \tag{1}$$

The extension operator is constructed through a process of localization and flattening. It is evident that the crucial part of the extension rests on the construction in a neighbourhood of the boundary of M. For this certain regularity assumptions on the boundary of M are imposed. This, together with introduction of the notation and terminology, is the subject of the first section. The extension theorem is proved in the next section. Here we construct a local extension with the help of the right inverse of the trace operator, a similar approach to that used in [2, Theorem 9.4.3]. However,

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in order to obtain control of the propagation of the support, a certain modification is proposed. The main idea is to use a convolution construction of the right inverse, see, e.g., [3, Lemma 2.5.6], which possesses a controllable propagation of the support. This will allow us to obtain the required control (1) of the propagation.

1. Preliminaries

Let Ω be an open set in $\mathbb{R}^n$ with boundary Γ and let M be in relative topology an open subset of Γ. We denote by $\bar{M}$ the closure of M and set $\partial M = \bar{M} \backslash M$ and $N = \Gamma \backslash \bar{M}$. Further, we denote by $d_M(x)$ the distance of the point $x \in \mathbb{R}^n$ from the set M and introduce the following spaces: $\mathscr{D}(\Omega)$, the space of real (or complex) functions on $\mathbb{R}^n$ that are infinitely differentiable and have a compact support contained in Ω; $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $p > 1$, the Sobolev space of functions $u(x)$ in $L^p(\Omega)$ such that all their weak derivatives $D^\alpha u$ of orders $|\alpha| \leqslant m$ belong to $L^p(\Omega)$. For sufficiently regular domain Ω we denote by γ the trace operator

$$\gamma\colon W^{m,p}(\Omega) \to \prod_{j=0}^{m-1} W^{m-j-1/p,p}(\Gamma)$$

and by $W^{m,p}_M(\Omega)$ the space

$$W^{m,p}_M(\Omega) = \{u : u \in W^{m,p}(\Omega) \text{ and } \gamma u|_M = 0\}.$$

For definition and existence of the trace operator we refer to [3]. Restriction of the trace operator to the subset M will be denoted by γ_M.

For a given orthogonal coordinate system in $\mathbb{R}^n$ we denote by $\mathbf{e}_k$ the unit basis vector in the direction of the axis x_k and by D_k partial differentiation with respect to the variable x_k. Further, the unit n-dimensional hypercube is denoted by Q, while Q' and Q'' stand for $(n-1)$- and $(n-2)$-dimensional hypercubes, respectively. Hypercubes with sides of length 1/2 are further denoted by a subscript 1/2. By $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ are denoted the half spaces with positive and negative x_n coordinate, respectively. Finally, we define $Q^+ = Q \cap \mathbb{R}^n_+$ and $Q^- = Q \cap \mathbb{R}^n_-$; similarly we also define $Q^+_{1/2}$ and $Q^-_{1/2}$, and introduce the space

$$W^{m,p}_0(Q^-_{1/2} \cup Q') = \{u : u \in W^{m,p}(Q_{1/2}) \text{ and } \operatorname{supp} u \subset Q^-_{1/2} \cup Q'\}.$$

We also need a notion of the pyramid P and an oblique pyramid P_o defined for any $o \in \mathbb{R}$. The former is defined as

$$P = \{y : y \in \mathbb{R}^n,\ 0 < y_n < 1 \text{ and } |y_i| < 1 - y_n \text{ for } i \in \{1, ..., n-1\}\},$$

while the latter is

$$P_o = \{ y = (y'', y_{n-1}, y_n) : y \in \mathbb{R}^n \text{ and } (y'', y_{n-1} - oy_n, y_n) \in P \}.$$

Now after the notation has been introduced, we are ready to specify regularity conditions upon ∂M.

DEFINITION 1. Let $\bar{\Omega} \subset \mathbb{R}^n$ be an n-dimensional compact C^m $(m \geq 1)$ manifold with boundary Γ and let $\mathscr{A}$ be a corresponding atlas on $\bar{\Omega}$. We say that a compact set $\bar{M} \subset \Gamma$ has the *uniform cone property* (u.c.p. for short) *relative to* $(\Gamma, \mathscr{A})$ if for every $x \in \partial M$ there exist an open neighbourhood U_x of x in $\mathbb{R}^n$, a chart $(\alpha, U) \in \mathscr{A}$, and an $(n-1)$-dimensional finite cone $C_{x,\alpha} \subset \mathbb{R}^{n-1} \times \{0\}$ such that $U_x \subset U$ and

$$\xi + C_{x,\alpha} \subset \alpha(M \cap U) \qquad \text{for any} \quad \xi \in \alpha(U_x) \cap (\mathbb{R}^{n-1} \times \{0\}).$$

We note first that the definition does not depend on a particular choice of the atlas $\mathscr{A}$; i.e., if M has the u.c.p. relative to $(\Gamma, \mathscr{A})$, then M has also the u.c.p. relative to $(\Gamma, \mathscr{A}')$ for any atlas $\mathscr{A}'$ equivalent to $\mathscr{A}$. Indeed, if $(\beta, V) \in \mathscr{A}'$ and $x \in U \cap V$, then $\beta\alpha^{-1} : \alpha(U \cap V) \to \beta(U \cap V)$ is a C^1 diffeomorphism and consequently for any $(n-1)$-dimensional cone $\eta_0 + C_\alpha \subset \alpha(U \cap V) \cap (\mathbb{R}^{n-1} \times \{0\})$, with the vertex at η_0, there exists a cone $\beta\alpha^{-1}(\eta_0) + C_\beta \subset \beta(U \cap V) \cap (\mathbb{R}^{n-1} \times \{0\})$, with the vertex at $\beta\alpha^{-1}(\eta_0)$, such that $\beta\alpha^{-1}(\eta_0) + C_\beta \subset \beta\alpha^{-1}(\eta_0 + C_\alpha)$. Moreover, since $\beta\alpha^{-1}$ is of class C^1, there exists an open neighbourhood W' of η_0 in $\mathbb{R}^{n-1} \times \{0\}$ such that $\beta\alpha^{-1}(\eta) + C_\beta \subset \beta\alpha^{-1}(\eta + C_\alpha)$, whenever $\eta \in W'$. Therefore, for a sufficiently small neighbourhood $V' \subset \beta(U \cap V)$ of $\beta(x)$ in $\mathbb{R}^n$, there exists a cone $C_{x,\beta}$ such that

$$\xi + C_{x,\beta} \subset \beta(M \cap V) \qquad \text{for any} \quad \xi \in V' \cap (\mathbb{R}^{n-1} \times \{0\})$$

and this proves that M has the u.c.p. relative to $(\Gamma, \mathscr{A}')$.

Remark 1. For a Lipschitz manifold the u.c.p. relative to Γ is not necessarily an intrinsic property of the manifold. Indeed, there is a known example, see, e.g., [4, pp. 8–9], of a bi-Lipschitz mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps a domain with the uniform cone property into a domain which does not posses the uniform cone property.

It is well known, see, e.g., [4, Theorem 1.2.2.2], that for any open subset $\Omega \subset \mathbb{R}^n$ the uniform cone property implies that the boundary of Ω is Lipschitz. Similarly, it follows from Definition 1 that there exist a neighbourhood W' of $\alpha(x)$ in $\mathbb{R}^{n-1} \times \{0\}$ and a new set of orthogonal coordinates $(y, ..., y_{n-1})$ such that

(i) W' is a hypercube in the new coordinates

$$W' = \{(y, ..., y_{n-1}) : -a_i < y_i < a_i,\ 1 \leqslant i \leqslant n-1\},$$

(ii) there exists a Lipschitz continuous function f defined in

$$W'' = \{(y, ..., y_{n-2}) : -a_i < y_i < a_i,\ 1 \leqslant i \leqslant n-2\}$$

and such that

$$|f(y'')| \leqslant a_{n-1}/2 \text{ for any } y'' \in W'',$$
$$\alpha(M \cap U) \cap W' = \{y' = (y'', y_{n-1}) : y' \in W' \text{ and } y_{n-1} > f(y'')\},$$
$$\alpha(\partial M \cap U) \cap W' = \{y' = (y'', y_{n-1}) : y' \in W' \text{ and } y_{n-1} = f(y'')\}.$$

It is evident that with a proper scaling of the charts (α, U) we can assume that W' is the $(n-1)$-dimensional hypercube $Q'_{1/2}$. Thus we have obtained:

Proposition 1. *Let $\bar{\Omega}$ be an n-dimensional compact C^m $(m \geqslant 1)$ manifold with boundary Γ, let $\mathscr{A}$ be a corresponding atlas on $\bar{\Omega}$, and let a compact set $\bar{M} \subset \Gamma$ have the u.c.p. relative to $(\Gamma, \mathscr{A})$. Then there exists an equivalent atlas $\mathscr{A}'$ such that for any $x \in \partial M$ there is a chart $(\alpha, U) \in \mathscr{A}'$ such that $x \in U$, $\alpha(U)$ is a n-dimensional hypercube $Q_{1/2}$, $\alpha(U \cap \Omega) = Q^-_{1/2}$, and $\alpha(M \cap U)$ is an $(n-1)$-dimensional epigraph of a Lipschitz continuous function f defined in $Q''_{1/2}$.*

Let us now introduce the following notation. For each chart $(\alpha, U) \in \mathscr{A}'$ we define $M'_\alpha = \alpha(M \cap U)$, $N'_\alpha = Q'_{1/2} \backslash M'_\alpha$, and $\partial M'_\alpha = \alpha(\partial M \cap U)$. Since, due to the localization, only one chart will be considered simultaneously, we shall, in order to simplify notation, drop the subscript α. Further, we denote by $d_{M'}(x)$, $d_{N'}(x)$, and $d_{\partial M'}(x)$ the distance to the point $x \in \mathbb{R}^n$ from the set M', N', and $\partial M'$, respectively.

In the local coordinates N' can be expressed as

$$N' = \{y' = (y'', y_{n-1}) : y'' \in Q''_{1/2} \text{ and } y_{n-1} < f(y'')\}, \tag{2}$$

while $\partial M'$ is a graph of the Lipschitz continuous function f. It is known, see, e.g., [5, Corollary 4.8], that on a suitably chosen set $W' \subset Q'_{1/2}$ the distance $d_{\partial M'}(x)$ is equivalent to the distance in the direction of the y_{n-1} axis. This means that there exists a constant κ, which depends only on f, such that for any $y' \in W'$

$$\frac{1}{\kappa} |y_{n-1} - f(y'')| \leqslant d_{\partial M'}(y') \leqslant \kappa\, |y_{n-1} - f(y'')|. \tag{3}$$

Evidently, the charts (α, U) can be arranged so that $W' = Q'_{1/2}$ and consequently we assume that this is already the case. We note also that for $y' \in M'$ or $y' \in N'$ we have $d_{\partial M'}(y') = d_{N'}(y')$ and $d_{\partial M'}(y') = d_{M'}(y')$, respectively.

We conclude this section with the following lemma:

LEMMA 1. *If $t \geqslant \kappa\delta$ then*

$$Q'_{1/2} \cap (\{x' : \operatorname{dist}(x', N') < \delta\} - t\mathbf{e}_{n-1}) \subset N'.$$

Proof. Let $y' \in \{x' : \operatorname{dist}(x', N') < \delta\}$. For $y' \in N'$ the statement of the lemma is obvious, so let us assume $y' \notin N'$. Then $\operatorname{dist}(y', N') = d_{\partial M'}(y')$ and from (3) it follows that

$$y_{n-1} - f(y'') < \kappa\delta$$

and

$$y_{n-1} - t < \kappa\delta - t + f(y'') < f(y''),$$

which together with (2) proves the lemma.

2. CONSTRUCTION OF THE EXTENSION OPERATOR

LEMMA 2. *For any real number a there exist an oblique pyramid P_a and a right inverse σ_a of the trace operator $\gamma \in \mathscr{L}(W^{m,p}(P_a), \prod_{j=0}^{m-1} W^{m-j-1/p,p}(Q'))$ such that if $\gamma_{M'}\varphi = 0$, then*

$$\operatorname{supp} \sigma_a \varphi \cap \{M' \times \{0\} + \mathbb{R}_+ \cdot (a\mathbf{e}_{n-1} + \mathbf{e}_n)\} = \varnothing. \tag{4}$$

Proof. Let us denote by ψ_j a function from $W^{m-j-1/p,p}(Q')$ such that $\psi_j = 0$ on M'. We note, see e.g., [3, Lemma 2.5.6], that the convolution operator

$$\sigma_j \psi_j(y', y_n) = \frac{1}{j!}\, y_n^j \int_{|z'|<1} \theta(z')\, \psi_j(y' + y_n z')\, dz' \tag{5}$$

is a bounded linear operator from $W^{m-j-1/p,p}(Q')$ to $W^{m-j,p}(P)$ and

$$\gamma_k \sigma_j \psi_j = \begin{cases} 0 & j > k \\ \psi_j & j = k. \end{cases} \tag{6}$$

Now it follows from (5) and Lemma 1 that the influence of N' propagates in the direction of the axis y_{n-1}, where y_n acts as the time, with a speed not exceeding κ, and this means that

$$\operatorname{supp} \sigma_j \psi_j \cap \{M' \times \{0\} + \mathbb{R}_+ (\kappa \mathbf{e}_{n-1} + \mathbf{e}_n)\} = \varnothing.$$

Thus in order to prove (4) we transform the pyramid P through the transformation

$$\tau_{-o}: y=(y'', y_{n-1}, y_n) \rightarrow (y'', y_{n-1}+oy_n, y_n)$$

into an oblique pyramid P_o. Evidently $\tau_{-o}^{-1}=\tau_o$ and if $u \in W^{m-j,p}(P)$, then $\tau_o^* u := u\tau_o \in W^{m-j,p}(P_o)$. Therefore, for $o=a-\kappa$ we obtain

$$\operatorname{supp} \tau_o^* \sigma_j \psi_j \cap \{M' \times \{0\} + \mathbb{R}_+(a\mathbf{e}_{n-1}+\mathbf{e}_n)\} = \varnothing. \tag{7}$$

Now it remains to construct, on the basis of the operators $\tau_o^* \sigma_j$, the right inverse σ. First we note that for $u \in W^{m-j,p}(P)$ we have

$$D_n \tau_o^* u = \tau_o^*(-oD_{n-1}u + D_n u) = \tau_o^* D_o u$$

and

$$D_n^k \tau_o^* u = \tau_o^* D_o^k u = \tau_o^* \sum_{i=0}^{k} \binom{k}{i} (-oD_{n-1})^{k-i} D_n^i u.$$

Above we introduced the notation $D_o = -oD_{n-1} + D_n$. Thus for $\psi_j \in W^{m-j-1/p,p}(Q')$ and $k \in \{0, ..., m-1\}$ we have

$$\gamma_k \tau_o^* \sigma_j \psi_j = \tau_o^* D_o^k \sigma_j \psi_j |_{y_n=0} = \sum_{i=0}^{k} \binom{k}{i} (-oD_{n-1})^{k-i} \gamma_i \sigma_j \psi_j$$

and according to (6)

$$\gamma_k \tau_o^* \sigma_j \psi_j = \begin{cases} 0 & j>k \\ \sum_{i=j}^{k} \binom{k}{i}(-oD_{n-1})^{k-i} \gamma_i \sigma_j \psi_j & j \leqslant k. \end{cases} \tag{8}$$

We note that all differentiations in (8) are well defined and thus consequently $\gamma_k \tau_o^* \sigma_j \psi_j \in W^{m-k-1/p,p}(Q')$. Now we set

$$\sigma\psi = \sum_{j=0}^{m-1} \tau_o^* \sigma_j \psi_j,$$

where $\psi = (\psi_0, ..., \psi_{m-1}) \in W := \prod_{j=0}^{m-1} W^{m-j-1/p,p}(Q')$. Obviously we have $\sigma \in \mathscr{L}(W, W^{m,p}(P_o))$ and it follows from (8) that

$$\gamma_k \sigma\psi = \sum_{j=0}^{m-1} \gamma_k \tau_o^* \sigma_j \psi_j = \sum_{j=0}^{k} \sum_{i=j}^{k} \binom{k}{i} (-oD_{n-1})^{k-i} \gamma_i \sigma_j \psi_j. \tag{9}$$

We now prove that there exist a bounded linear operator $\beta \in \mathscr{L}(W, W)$

such that if $\psi = \beta\varphi$, then $\gamma_k \sigma \psi = \varphi_k$ and $\operatorname{supp} \psi \subset \operatorname{supp} \varphi$. For this we rewrite (9) into

$$\gamma_k \sigma \psi = \psi_k + \sum_{j=0}^{k-1} \gamma_k \sigma_j \psi_j + \sum_{j \leqslant i < k} \binom{k}{i} (-oD_{n-1})^{k-1} \gamma_i \sigma_j \psi_j.$$

The expression determines, through the formula $\gamma_k \sigma \psi = \varphi_k$, the operator β in a recursive way. Evidently $\beta \in \mathscr{L}(W, W)$ and it remains to check whether $\operatorname{supp} \psi \subset \operatorname{supp} \varphi$. From (5) we obtain for $i \geqslant j$

$$\begin{aligned} \gamma_i \sigma_j \psi_j(y') &= \lim_{y_n \to 0} \sum_{|\lambda| = i} c_\lambda \int_{|z'| < 1} \theta(z')\, z'^\lambda D^\lambda \psi_j(y' + y_n z')\, dz' \\ &= \sum_{|\lambda| = i} c'_\lambda D^\lambda \psi_j(y') \end{aligned}$$

for certain constants c_λ and c'_λ. It follows from the above expression that $\operatorname{supp} \gamma_i \sigma_j \psi_j \subset \operatorname{supp} \psi_j$ and therefore

$$\operatorname{supp} \psi_k \subset \bigcup_{j=0}^{k-1} \operatorname{supp} \psi_j \cup \operatorname{supp} \varphi_k$$

and since $\psi_0 = \varphi_0$ we have proved

$$\operatorname{supp} \psi \subset \operatorname{supp} \varphi.$$

Thus $\sigma_a = \sum_{j=0}^{m-1} \tau_o^* \sigma_j \beta_j$ is the required right inverse and the proof of the lemma is completed.

LEMMA 3. *For each positive real number b there exists an extension operator $\Pi \in \mathscr{L}(W_0^{m,p}(Q_{1/2}^- \cup Q'),\ W^{m,p}(\mathbb{R}^n))$ such that if $\gamma_{M'} u^- = 0$, $u^- \in W_0^{m,p}(Q_{1/2}^- \cup Q')$, then*

$$\operatorname{supp} \Pi u^- \cap \mathbb{R}_+^n \subset \{ y : d_{N'}(y) < b \cdot d_{\partial M'}(y) \}. \tag{10}$$

Proof. We take an arbitrary $u^- \in W_0^{m,p}(Q_{1/2}^- \cup Q')$ and extend it by zero to $u^- \in W_0^{m,p}(Q^- \cup Q')$. Let $a := -(1 + \kappa/b)$ and denote by σ_a and P_o the right inverse and the oblique pyramid from Lemma 2. This allows us to construct for $\varphi := \gamma u^-$ a function $u^+ = \sigma_a \varphi \in W^{m,p}(P_o)$. Then $\gamma u^+ = \gamma u^-$ and according to the matching lemma, see, e.g., [4, Theorem 1.7.1], a function

$$u(y', y_n) = \begin{cases} u^- & y_n < 0 \\ u^+ & y_n > 0 \end{cases}$$

belongs to $W^{m,p}(Q^- \cup P_o)$. Now we denote by $V \subset Q^- \cup P_o$ an open set which strictly contains $Q_{1/2}$. Further, let us introduce a function $\omega \in \mathscr{D}(V)$ such that $\omega \equiv 1$ on $Q_{1/2}^-$. Evidently ω depends only on b and therefore

$$\Pi u^- = \begin{cases} u^- & y_n < 0 \\ \omega \sigma_a \gamma u^- & y_n > 0 \end{cases}$$

defines the extension operator $\Pi \in \mathscr{L}(W_0^{m,p}(Q_{1/2}^- \cup Q'), W^{m,p}(\mathbb{R}^n))$.

It remains to prove (10) for $\gamma_{M'} u^- = 0$. For this we note that $d^2_{\partial M'}(y', y_n) = y_n^2 + d^2_{\partial M'}(y', 0)$ and since $a < -1$, it follows from (4) that for any $y = (y'', y_{n-1}, y_n) \in Q^+ \cap \operatorname{supp} \Pi u^-$ we have $d^2_{N'}(y', y_n) = y_n^2$. Moreover,

$$\frac{y_n}{|f(y'') - y_{n-1}|} < \frac{1}{|a| - 1}$$

and after the estimate

$$\frac{1}{(|a|-1)^2} > \frac{d^2_{N'}(y)}{\kappa^2 d^2_{\partial M'}(y', 0)} \geqslant \frac{d^2_{N'}(y)}{\kappa^2 (d^2_{\partial M'}(y', 0) + y_n^2)}$$

we obtain

$$d_{N'}(y) < \frac{\kappa}{|a| - 1} d_{\partial M'}(y) = b \cdot d_{\partial M'}(y).$$

THEOREM 1. *Let $\bar{\Omega}$ be an n-dimensional compact C^m $(m \geqslant 1)$ manifold and let a compact set $\bar{M}$ have the u.c.p. relative to Γ. Then for each positive real number c there exist an extension operator $\Pi \in \mathscr{L}(W_M^{m,p}(\Omega), W^{m,p}(\mathbb{R}^n))$ such that*

$$\operatorname{supp} \Pi u \backslash \Omega \subset \{x : d_N(x) < c \cdot d_{\partial M}(x)\}. \tag{11}$$

Proof. Let $\{U_i\}_{i \in I}$ be a finite open covering of Γ consisting of the open sets referred to in Preliminaries and let us define a subset $I_1 \subset I$ such that

$$i \in I_1 \Leftrightarrow U_i \cap \partial M \neq \emptyset.$$

Further we set $\varepsilon_0 = \operatorname{dist}(\partial M, \Omega \backslash \bigcup \{U_i : i \in I_1\})$ and introduce an open set $U_0 \subset\subset \Omega$ such that $\{U_i\}_{i \in I \cup \{0\}}$ is an open covering of Ω. Since $\bar{\Omega}$ is compact, we may find a positive number ε_1 such that for any $x \in \Omega$ there exists $i \in I \cup \{0\}$ such that $B(x, \varepsilon_1) \subset U_i$. Here we denote by $B(x, \varepsilon_1)$ an open ball in in $\mathbb{R}^n$ of radius ε_1 centered at x. Further, let $\varepsilon_2 = \operatorname{dist}(\Gamma, U_0)$ and choose $\varepsilon = \min\{c\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Obviously $\Omega \subset \bigcup_{x \in \Omega} B(x, \varepsilon)$ and since $\bar{\Omega}$ is compact $\Omega \subset \bigcup_{j \in J} B(x_j, \varepsilon)$ for a certain finite index set J. We write $B_j = B(x_j, \varepsilon)$ and denote by $\{\varphi_j\}$ a partition of unity subordinated to

$\{B_j\}_{j\in J}$. Now we take an arbitrary $u\in W^{m,p}_M(\Omega)$ and write $u_j=u\varphi_j$. For each $j\in J$ there are two possibilities: $B_j\subset U_i$ either for $i\in I_1$ or for $i\notin I_1$. In the latter case we extend $u_j\in W^{m,p}(B_j\cap\Omega)$ in a standard way into a function $\hat{u}_j=\Pi_j u\in W^{m,p}_0(B_j)\subset W^{m,p}(\mathbb{R}^n)$. It readily follows from the construction of ε that

$$\operatorname{supp}\hat{u}_j\backslash\Omega\subset\{x: d_N(x)<c\cdot d_{\partial M}(x)\}$$

and so it remains to consider the former case $i\in I_1$.

For this we note the following properties of bi-Lipschitz mappings. Let F be a bi-Lipschitz mapping between open sets U, $V\subseteq\mathbb{R}^n$ and let L be an arbitrary subset of U. Then, as can be easily verified,

$$d_{F(L)}(F(x))\leqslant K_1 d_L(x),\qquad x\in U$$

and

$$d_{F^{-1}F(L)}(F^{-1}(y))\leqslant K_2 d_{F(L)}(y),\qquad y\in V,$$

where K_1 and K_2 are Lipschitz constants of F and F^{-1}, respectively. Since $i\in I_1$, $u_j\in W^{m,p}_M(U_i\cap\Omega)$ and after C^m coordinate transformation α_i of $U_i\cap\Omega$ onto $Q^-_{1/2}$, we have $u'_j:=\alpha_i u_j\in W^{m,p}_{M'}(Q^-_{1/2})$ and $\operatorname{supp}u'_j\subset Q^-_{1/2}\cup Q'$. Now, let us as above denote by K_1 and K_2 the Lipschitz constants of α_i and α_i^{-1} and let us set $b=c/(K_1K_2)$. For this value of b, after applying Lemma 3, we obtain the extension $\Pi'_j u'_j=:\hat{u}'_j\in W^{m,p}(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{u}'_j\cap\mathbb{R}^n_+\subset\{y: d_{N'}(y)<b\cdot d_{\partial M'}(y)\}$. Therefore, $\hat{u}_j=\Pi_j u_j:=\alpha^{-1}\hat{u}'_j$ defines the extension $\Pi_j\in\mathscr{L}(W^{m,p}_M(B_j\cap\Omega),\ W^{m,p}(\mathbb{R}^n))$.

Through the local extension operators Π_j now follows the existence of the global extension operator Π and it remains to see whether already holds (11). Indeed, if $x\in\operatorname{supp}\Pi_j u\backslash\Omega$, then $\alpha_i(x)\in\operatorname{supp}\hat{u}'_j\cap\mathbb{R}^n_+$ and

$$\begin{aligned} d_N(x)&=d_{\alpha_i^{-1}(N')}(\alpha_i^{-1}\alpha_i(x))\leqslant K_2 d_{N'}(\alpha_i(x))\\ &\leqslant\frac{c}{K_1K_2}K_2 d_{\partial M'}(\alpha_i(x))=\frac{c}{K_1}d_{\alpha_i(\partial M)}(\alpha_i(x))=c\cdot d_{\partial M}(x)\end{aligned}$$

and this concludes the proof.

From the preceding discussion easily follows the next corollary:

COROLLARY 1. *If a compact set $\bar{M}$ has the u.c.p. relative to Γ, then for any cone C such that $(x+C)\cap\Omega=\varnothing$ for every $x\in M$, there exists an extension operator Π: $W^{m,p}_M(\Omega)\to W^{m,p}(\mathbb{R}^n)$ such that* $(\operatorname{supp}\Pi u)\cap(x+C)=\varnothing$ *for all $x\in M$.*

Remark 2. It can be readily seen that Theorem 1 holds also for

$W^{1,p}(\Omega)$ and a Lipschitz manifold $\bar{\Omega}$ if there exists an atlas $\mathscr{A}$ with charts that possess all required properties from Preliminaries.

With the help of Theorem 1 can easily be deduced, in the standard way, see, e.g., [2, Theorem 9.5.4], the following density theorem, which is just a special case of a more general result, see [6]. However, it should be noted that our result does not depend on a particular choice of the covering of Γ.

THEOREM 2. *Let $\bar{\Omega}$ be an n-dimensional C^m manifold and let a compact set $\bar{M} \subset \Gamma$ have the u.c.p. relative to Γ. Then the space*

$$C_M^\infty(\Omega) = \{u : u \in C^\infty(\bar{\Omega}) \text{ and } \operatorname{supp} u \cap M = \varnothing\}$$

is a dense subset of $W_M^{m,p}(\Omega)$.

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